

Testing Unimodular Gravity

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Abstract: We consider models of gravitation that are based on unimodular general coordinate transformations (GCT). These transformations include only those which do not change the determinant of the metric. We treat the determinant as a separate field which transforms as a scalar under unimodular GCT. We consider a class of such theories. In general, these theories do not transform covariantly under the full GCT. We characterize the violation of general coordinate invariance by introducing a new parameter. We show that the theory is consistent with observations for a wide range of this parameter. This parameter may serve as a test for possible violations of general coordinate invariance. We also consider the cosmic evolution within the framework of these models. We show that in general we do not obtain consistent cosmological solutions if we assume the standard cosmological constant or the standard form of non-relativistic matter. We propose a suitable generalization which is consistent with cosmology. We fit the resulting model to the high redshift supernova data. We find that we can obtain a good fit to this data even if include only a single component, either cosmological constant or non-relativistic matter.

1 Introduction

The idea of unimodular gravity was first proposed by Anderson and Finkelstein [1]. It is closely related to an earlier proposal by Einstein [2]. In Ref. [1] the authors argued that it is natural to split the metric into two separate pieces, its determinant g and the remaining metric whose determinant is fixed to unity. In their proposal the determinant of the metric is not a field and hence not a dynamical variable. Later, it has been suggested that since the determinant is not a field in this theory, the cosmological constant term does not appear in the action. Thus this might help in solving the fine tuning problem of the cosmological constant. However, as nicely explained in the review paper by Weinberg [3], the problem is not really solved. The full action still has general coordinate invariance (GCI). Using this fact, Weinberg [3] shows that one again recovers the standard Einstein's equations. The only difference is that the cosmological constant now appears as an integration constant rather than a term in the action. The trace of the energy

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momentum tensor still contributes to the cosmological constant and the fine tuning problem is present.

Unimodular gravity has been pursued in detail in many papers [4–14], which have considered its application to the problem of the cosmological constant and its quantization. An interesting generalization is to consider the determinant as a dynamical field, as suggested by Zee [5] and Buchmuller and Dragon [6]. In this case the determinant effectively acts as a scalar field since one only demands invariance under unimodular general coordinate transformations (GCT). Hence one may include the determinant as a scalar field in the action [3, 6] and allow the full GCI to be broken. Such a procedure has been pursued in some papers in the literature [6, 15–19]. In the present paper we study some implications of this proposal. We study a model which represents a minimal generalization of the Einstein’s gravity and hence can be utilized to test the GCI. We introduce a parameter, which characterizes the violation of GCI and whose value may be determined or constrained observationally. For this purpose we study the cosmological implications of this model as well as its spherically symmetric solution in vacuum. We also use this model to fit the high z supernova data. The precise model we consider has not been investigated so far in the literature.

2 Models of Unimodular Gravity

We propose to split the standard metric as follows,

$$g_{\mu\nu} = \chi^2 \bar{g}_{\mu\nu}, \quad (1)$$

where we impose the following constraint on the determinant of $\bar{g}_{\mu\nu}$,

$$\bar{g} = \det[\bar{g}_{\mu\nu}] = f(x). \quad (2)$$

Here $f(x)$ is some specified function of the space-time coordinates. In literature this has generally been taken to be unity. However if we fix it as such then it excludes even the Lorentz metric in spherical coordinates. It seems better to generalize this constraint to at least allow $f(x)$ to be the determinant of the Lorentz metric, in what ever coordinates we choose to express it. We point out that \bar{g} is not a dynamical variable in this theory. Eq. 1 implies,

$$g^{\mu\nu} = \frac{1}{\chi^2} \bar{g}^{\mu\nu}. \quad (3)$$

We shall demand that our theory is invariant only under unimodular GCT. Under these transformations,

$$x^\mu \rightarrow x'^\mu \quad (4)$$

such that the Jacobian is unity, *i.e.*,

$$\det \left(\frac{\partial x'^\mu}{\partial x^\nu} \right) = 1. \quad (5)$$

The determinant of $g_{\mu\nu}$ and hence the field χ behaves as a scalar under these transformations. We treat χ as an independent scalar field. The basic quantities such as the connection, curvature tensor etc. split naturally into a function of $\bar{g}_{\mu\nu}$ and χ . We find,

$$\Gamma_{\alpha\beta}^\mu = \bar{\Gamma}_{\alpha\beta}^\mu + \tilde{\Gamma}_{\alpha\beta}^\mu, \quad (6)$$

where $\bar{\Gamma}_{\alpha\beta}^{\mu}$ is the connection computed using the metric $\bar{g}_{\mu\nu}$ and

$$\tilde{\Gamma}_{\alpha\beta}^{\mu} = \bar{g}_{\beta}^{\mu} \partial_{\alpha} \ln \chi + \bar{g}_{\alpha}^{\mu} \partial_{\beta} \ln \chi - \bar{g}_{\alpha\beta} \partial^{\mu} \ln \chi. \quad (7)$$

We point out that all raising and lowering of indices is done using the metric $\bar{g}_{\mu\nu}$. Furthermore all covariant derivatives are defined with respect to this metric. We see from Eq. 7 that $\tilde{\Gamma}_{\alpha\beta}^{\mu}$ behaves as a tensor under unimodular GCT. The Ricci curvature tensor may also be written as,

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \tilde{R}_{\mu\nu}, \quad (8)$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor computed by using the metric $\bar{g}_{\mu\nu}$ and

$$\tilde{R}_{\mu\nu} = -\tilde{\Gamma}_{\mu\nu;\alpha}^{\alpha} + \tilde{\Gamma}_{\mu\alpha;\nu}^{\alpha} - \tilde{\Gamma}_{\beta\alpha}^{\alpha} \tilde{\Gamma}_{\mu\nu}^{\beta} + \tilde{\Gamma}_{\beta\nu}^{\alpha} \tilde{\Gamma}_{\mu\alpha}^{\beta}. \quad (9)$$

We find

$$\tilde{R}_{\mu\nu} = 2(\ln \chi)_{;\mu;\nu} + \bar{g}_{\mu\nu}(\ln \chi)_{;\alpha}^{\alpha} - 2\partial_{\mu} \ln \chi \partial_{\nu} \ln \chi + 2\bar{g}_{\mu\nu} \partial_{\beta} \ln \chi \partial^{\beta} \ln \chi. \quad (10)$$

Contracting both $\bar{R}_{\mu\nu}$ and $\tilde{R}_{\mu\nu}$ with the metric $\bar{g}^{\mu\nu}$ we obtain the scalars \bar{R} and \tilde{R} respectively. Hence we can express the Ricci scalar R as

$$R = g^{\mu\nu} R_{\mu\nu} = \frac{1}{\chi^2} (\bar{R} + \tilde{R}). \quad (11)$$

This gives,

$$\tilde{R} = 6(\ln \chi)_{;\mu}^{\mu} + 6\partial_{\mu} \ln \chi \partial^{\mu} \ln \chi. \quad (12)$$

We may now express the Einstein action in terms of quantities computed by the metric $\bar{g}_{\mu\nu}$ and χ ,

$$S_E = \int d^4x \sqrt{-\bar{g}} \frac{1}{16\pi G} [\chi^2 \bar{R} + \chi^2 \tilde{R}]. \quad (13)$$

Expressing \tilde{R} explicitly in terms of χ we find, after an integration by parts,

$$S_E = \int d^4x \sqrt{-\bar{g}} \frac{1}{16\pi G} [\chi^2 \bar{R} - \xi \partial_{\alpha} \chi \partial^{\alpha} \chi], \quad (14)$$

where the parameter $\xi = 6$. This action, given in Eq. 14, is exactly the same as the standard Einstein's action as long as $\xi = 6$. However we treat ξ as an arbitrary parameter to be fitted to experimental data. For other values of ξ , the action in Eq. 14 will respect only unimodular GCI but not the full symmetry group. Hence the parameter ξ also provides a test of the validity of the GCI.

The action given in Eq. 14 has to be supplemented by the matter action. We discuss some simple examples of matter action in the next section. Here we clarify that our model preserves most of the structure of Einstein's theory. Hence we still have the freedom to choose a locally inertial coordinate system which is valid in a small neighborhood of any point. This is because our metric $\bar{g}_{\mu\nu}$ has determinant equal to the determinant of the Lorentz metric. Hence we can transform it locally to the Lorentz metric by making a unimodular GCT. Indeed we may choose any coordinate system in our unimodular theory subject to the constrain that we can locally transform it to the Lorentz metric by making a unimodular GCT.

There are many possible ways to generalize the model such that it displays invariance under unimodular GCT but not the full GCI. We may, for example, include terms with any power of the field χ without breaking unimodular GCI [6, 15–19]. We may also introduce other terms which

violate GCI [20]. The model presented in Eq. 14 with $\xi \neq 6$ represents a minimal modification of the Einstein's gravity. The matter action may also be modified so as to display only unimodular GCI. We shall consider some simple examples of such modification in the next section. It may be interesting to consider the most general model within the framework of unimodular gravity. However such a model would introduce a large number of parameters and it appears better to first get some familiarity with these models in a simpler framework.

We point out that our theory is covariant under unimodular GCT. Hence the action and the field equations remain invariant under these transformations. Under the full GCT, the action and the equations of motion would change. However one may make the action look formally invariant under the full GCT by introducing a compensator field [15]. Let us assume that we define our unimodular theory in some coordinate system \bar{x} . One may, of course, choose any coordinate system which is related to \bar{x} by a unimodular transformation. Now let's make a general coordinate transformation, denoting the new coordinates by x . The compensator field is defined as, $C(x) \equiv D(x, \bar{x})$, which is the determinant of the transformation between \bar{x} and x [15]. In this case the action formally appears invariant under GCT [15, 19], with the introduction of an additional scalar field. This procedure may offer some advantages for some applications. However in the present case we prefer to work directly with the original variables without introducing the compensator field. Let us consider, for example, an application of our model to cosmology. Here the simplest metric to use is precisely a preferred frame in which we expect our dynamical equations to reduce to the unimodular equations. We discuss this in section 3. Hence it appears simpler to work directly with the original variables. Furthermore we point out that one can use the general coordinate system, with suitably modified action, only to compute observables which are invariant under GCT. However in a unimodular theory all quantities which are invariant under a limited unimodular transformations are physically observable. For quantities which are invariant under only the unimodular transformations and not the full GCT, the results will depend on the coordinate system, if we use the covariant construction by introducing a compensator field. Hence in this sense the unimodular theory is not completely equivalent to the fully covariant theory constructed by introducing a compensator field. For these reasons, here we do not introduce the compensator field and work directly with the original non-covariant theory.

2.1 Particle Trajectory Equation

The geodesic equation for particle motion can be expressed in terms of the variable χ by use of Eq. 6. We find,

$$\frac{d^2 x^\mu}{d\lambda^2} + \bar{\Gamma}_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} + \tilde{\Gamma}_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0. \quad (15)$$

The first two terms on the left hand side represent the usual terms due to motion of particle in gravitational field described by the metric $\bar{g}_{\mu\nu}$. The third term represents the additional contribution due to χ . Hence the scalar field χ simply provides an additional force which the particles experience. If we demand only unimodular GCI it is of course possible to generalize this equation. We may, for example, insert a parameter multiplying the last term on the left hand side of this equation. Limits on this parameter may be imposed observationally. In this paper we shall assume that this parameter is unity. In the next section we discuss cosmic evolution within the framework of our model. As we shall see this requires modification of the particle trajectory equation in some cases.

3 Cosmic Evolution

We next use our model to consider evolution of the universe. The unimodular gravitational equation of motion may be written as

$$-\chi^2 \left[\bar{R}_{\mu\nu} - \frac{1}{4} \bar{g}_{\mu\nu} \bar{R} \right] - \left[(\chi^2)_{;\mu;\nu} - \frac{1}{4} \bar{g}_{\mu\nu} (\chi^2)_{;\lambda}^{\lambda} \right] + \xi \left[\partial_\mu \chi \partial_\nu \chi - \frac{1}{4} \bar{g}_{\mu\nu} \partial^\lambda \chi \partial_\lambda \chi \right] = \frac{\kappa}{2} \left[T_{\mu\nu} - \frac{1}{4} \bar{g}_{\mu\nu} T^\lambda_\lambda \right], \quad (16)$$

where $\kappa = 16\pi G$. The equation of motion of the field χ gives,

$$2\chi \bar{R} + 2\xi \bar{g}^{\mu\nu} \chi_{;\mu;\nu} = \kappa T_\chi, \quad (17)$$

where T_χ includes all the contribution due to the coupling of χ with matter fields. We shall discuss examples of this below. Here we may point out that the energy-momentum tensor, $T_{\mu\nu}$, as defined in Eq. 16, need not satisfy the usual conservation law. It is simply the contribution given by matter fields to this equation. The curvature tensor does indeed satisfy the Bianchi identity but this does not necessarily imply a conservation law for $T_{\mu\nu}$. We may, however, obtain a conservation law by obtaining an expression for $\bar{R}_{\mu\nu} - \bar{g}_{\mu\nu} \bar{R}/2$ by using Eqs. 16 and 17. This will define a generalized energy momentum tensor which will satisfy the conservation law, since,

$$\left[\bar{R}_\nu^\mu - \frac{1}{2} \bar{g}_\nu^\mu \bar{R} \right]_{;\mu} = 0. \quad (18)$$

3.1 Cosmological Constant

We first consider the simple case of cosmological constant or vacuum energy. This term contributes to the action in the form

$$S_{vac} = \int d^4x \sqrt{-g} [-\chi^4 \Lambda], \quad (19)$$

which gives,

$$T_\chi = 4\chi^3 \Lambda. \quad (20)$$

However it does not contribute to Eq. 16. For simplicity we assume a spatially flat FRW metric,

$$g_{\mu\nu} = \chi^2(\eta) \bar{g}_{\mu\nu} \quad (21)$$

with

$$\bar{g}_{\mu\nu} = \text{diagonal}[1, -1, -1, -1]. \quad (22)$$

Here η denotes the conformal time. Our metric is essentially the same as the standard spatially flat Friedmann-Robertson-Walker (FRW) metric written in terms of conformal time. We do not consider more complicated metrics in the present paper. This implies $\bar{R} = 0$ and $\bar{R}_{\mu\nu} = 0$. Setting $\mu = \nu = 0$ in Eq. 16 we find

$$2\chi \frac{d^2\chi}{d\eta^2} + (2 - \xi) \left[\frac{d\chi}{d\eta} \right]^2 = 0. \quad (23)$$

Using Eqs. 17 and 20 we find

$$2\xi \frac{d^2\chi}{d\eta^2} = 4\kappa\chi^3 \Lambda \quad (24)$$

This equation along with Eq. 23 leads to,

$$\frac{d\chi}{d\eta} = \pm \sqrt{\frac{4\kappa\Lambda}{\xi(\xi-2)}} \chi^2. \quad (25)$$

For $\xi = 6$ this gives the standard solution obtained by using the Einstein's equations. For $\xi \neq 6$ we find that there does not exist any solution to these set of equations, Eq. 23 and Eq. 25. The only allowed solution is that $\chi = 0$. Hence we find a very interesting result that the vacuum dominated solution exists only for the standard case $\xi = 6$. This is interesting since it might provide us with a solution to the problem of fine tuning of the cosmological constant. A large cosmological constant here does not imply a rapidly expanding universe. A consistent solution may be obtained with a cosmological constant only if some other component, such as matter or radiation, also contributes to energy density. Alternatively we may generalize the action for the cosmological constant term, which is permissible within the framework our unimodular theory. We discuss this in the next subsection.

3.2 Generalized Cosmological Constant

We next determine whether it is possible to generalize the cosmological constant term such that it leads to consistent cosmology in the general case when ξ is different from 6. We assume an action of the form

$$S'_{vac} = \int d^4x \sqrt{-g} [-\chi^\delta \Lambda], \quad (26)$$

instead of Eq. 19. In Eq. 26 δ is a constant. We shall fix this constant by demanding that it leads to consistent cosmology. For the standard case of $\xi = 6$ we have $\delta = 4$. The unimodular gravitational equation of motion remains the same as Eq. 23 whereas the equation of motion for χ changes to,

$$2\xi \frac{d^2\chi}{d\eta^2} = \delta\kappa\chi^{(\delta-1)}\Lambda \quad (27)$$

Eliminating the second derivative between Eq. 23 and Eq. 27, we obtain

$$\frac{d\chi}{d\eta} = \pm \sqrt{\frac{\delta\kappa\Lambda}{\xi(\xi-2)}} \chi^{\delta/2}. \quad (28)$$

Demanding that this equation is consistent with Eq. 27 we obtain,

$$\delta = \xi - 2 \quad (29)$$

We next solve for $\chi(t)$ where t is the cosmic time such that $\chi d\eta = dt$. The expanding solution, corresponding to the positive sign in Eq. 28, is found to be

$$\chi(t) = \left[1 - \frac{6-\xi}{2} \sqrt{\frac{\kappa\Lambda}{\xi}} (t_0 - t) \right]^{2/(6-\xi)} \quad (30)$$

Here t_0 is the current time and we have set $\chi(t_0) = 1$.

It is useful to determine whether this vacuum component by itself can explain cosmological observations. We partially address this issue in the present paper by fitting this model to the high redshift supernova data. We next obtain the luminosity distance, d_L , in this model. The

luminosity distance is given by,

$$d_L = \frac{r}{\chi} = (z+1) \int_{t(z)}^{t_0} \frac{dt'}{\chi(t')}, \quad (31)$$

where t_0 is the current time and $t(z)$ is the time when the light left the source located at a redshift of z . We obtain

$$d_L = \frac{2(1+z)}{(\xi-4)H_0} \left[(1+z)^{(\xi-4)/2} - 1 \right] \quad (32)$$

where $H_0 = \sqrt{\kappa\Lambda/\xi}$ is the current value of the Hubble constant. The relation between luminosity distance and distance modulus μ is given by,

$$\mu = m - M = 5(\log_{10} d_L - 1). \quad (33)$$

We fit the supernova data using the compilation of 557 sources available in [21]. The resulting statistic χ^2 as a function of the parameter ξ is shown in Fig. 1. Here we have used the d_L given by Eq. 32 obtained by including only the vacuum contributions defined by the action, Eq. 26. We find that χ^2 attains its minimum value of 549.2 at $\xi = 4.76$. At this value of ξ the Hubble parameter is found to be 69.4 Km/(sec Mpc). Since χ^2 per degree of freedom is less than unity we find that this single component model also provides a good fit to the data. For comparison the standard Λ CDM model, which includes both cosmological constant and non-relativistic dark matter, leads to χ^2 equal to 542.8.

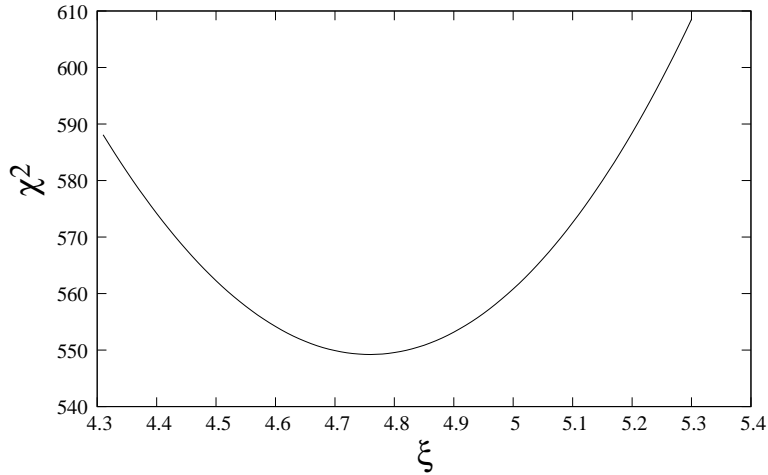


Figure 1: The χ^2 as a function of the parameter ξ for purely vacuum dominated Universe as defined by Eq. 26.

3.3 Radiation

We next solve our equations for a universe dominated by relativistic matter. We assume a scalar field with action given by,

$$S_{rad} = \int d^4x \sqrt{-\bar{g}} \frac{1}{2} \chi^2 \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \quad (34)$$

where the mass term for the scalar ϕ is assumed to be negligible. We identify energy momentum tensor as,

$$T_{\mu\nu} = \chi^2 \langle \partial_\mu \phi \partial_\nu \phi \rangle, \quad (35)$$

where the expectation value is taken in an appropriate thermal state corresponding to the temperature of the medium. Here we are treating ϕ as a quantum field and the fields $\bar{g}_{\mu\nu}$, χ etc. classically. In Eq. 35 the contribution to $T_{\mu\nu}$ is obtained by identifying the contribution from the matter fields to the unimodular gravitational equations of motion, Eq. 16. We do not get any contributions proportional to $\bar{g}_{\mu\nu}$, which are present in the standard energy momentum tensor. In any case, such contributions would cancel when we subtract the trace.

We can now relate $\langle \partial_\mu \phi \partial_\nu \phi \rangle$ to the usual definition of energy density. Let $\Theta_{\mu\nu}$ denote the standard energy momentum tensor in the Einstein's gravity. Then the standard expression for energy momentum tensor gives,

$$\Theta_{00} = \langle \partial_0 \phi \partial_0 \phi \rangle = g_{00} \Theta_0^0 = a^2(\eta) \rho, \quad (36)$$

where $a(\eta)$ is the standard FRW scale factor defined in terms of conformal time and ρ is the energy density of radiation. Here we have used the fact that $\Theta_0^0 = \rho$. Furthermore, we identify the scale factor $a(\eta)$ with χ . Hence we obtain,

$$\langle \partial_0 \phi \partial_0 \phi \rangle = \chi^2(\eta) \rho. \quad (37)$$

The contribution T_χ to Eq. 17 is given by

$$T_\chi = -\chi \langle \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \rangle. \quad (38)$$

In momentum space each particle will contribute a term proportional to $\bar{g}^{\mu\nu} P_\mu P_\nu$, where P_μ is its momentum, to this equation. Hence T_χ is equal to zero for relativistic matter.

We next solve the equations of motion setting $\bar{g}_{\mu\nu}$ equal to the Minkowski metric. The equation of motion for χ , Eq. 17, gives

$$\frac{d^2 \chi}{d\eta^2} = 0. \quad (39)$$

This equation implies that $d\chi/d\eta = \text{constant}$. The unimodular equations of motion, Eq. 16 gives

$$(\xi - 2) \left(\frac{d\chi}{d\eta} \right)^2 = \frac{2\kappa}{3} \chi^4 \rho. \quad (40)$$

This gives the standard equation for the scale parameter in terms of conformal time if we set $\xi = 6$. For other values it represents the generalization to unimodular gravity. Irrespective of the value of ξ we find that

$$\rho \propto \frac{1}{\chi^4}. \quad (41)$$

The value of $\xi > 2$ since for smaller values the equation gives complex solutions, which are physically not acceptable. For all values of $\xi > 2$, the evolution is the same as in the standard

Einstein's gravity up to an overall constant factor which may be absorbed in the initial condition on the energy density of the radiation field.

We may verify the relationship between ρ and χ , Eq. 41, directly from the action. For this purpose we define the scaled variable $\bar{\phi} = \chi\phi$. In terms of the scaled field we may write the action, after an integration by parts, as,

$$S_{rad} = \int d^4x \sqrt{-\bar{g}} \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} + \int d^4x \sqrt{-\bar{g}} \frac{1}{2} \bar{g}^{\mu\nu} \frac{\bar{\phi}^2}{\chi} \partial_\mu \partial_\nu \chi, \quad (42)$$

We assume that the background field χ is a slowly varying function of time. Hence in the adiabatic limit we may ignore its derivatives. Within this approximation we may write the action as

$$S_{rad} \approx \int d^4x \sqrt{-\bar{g}} \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi}, \quad (43)$$

The corrections to this are of order H^2/ω^2 , where H is the Hubble constant and ω the frequency of the radiation field. Hence for all frequencies of physical interest, this ratio is extremely small and can be safely neglected. We may now treat this as the standard massless free field in flat space-time since we have taken $\bar{g}_{\mu\nu}$ as the Lorentz metric. The expectation value of the Hamiltonian density, \mathcal{H} , is simply equal to the energy density E/V where E is the total energy and V the spatial volume. For any multi-particle eigenstate of the Hamiltonian this is equal to the sum of frequencies of all the particles divided by the volume of space. We point out that the conformal time, η , dependence of the free field solutions is given by $e^{-i\eta\omega}$. The spatial volume in the present flat space-time case is independent of χ . The frequencies, ω , are also independent of χ . Hence we find,

$$\langle \mathcal{H} \rangle \approx \left\langle \frac{\partial \bar{\phi}}{\partial \eta} \frac{\partial \bar{\phi}}{\partial \eta} \right\rangle \quad (44)$$

is independent of χ . This is true for all states since they can be written as linear superpositions of the eigenstates. We point out that the two terms involving the time and space derivatives in the Hamiltonian density contribute equally to the expectation value in Eq. 44. This implies that

$$\left\langle \frac{\partial \phi}{\partial \eta} \frac{\partial \phi}{\partial \eta} \right\rangle \propto \frac{1}{\chi^2} \quad (45)$$

Finally using Eq. 37 we obtain Eq. 41.

3.4 Non-relativistic Matter

For the non-relativistic matter we again use a scalar field to obtain the matter contributions to our equations of motion. We first use the standard covariant action decomposed in terms of the field χ and the metric $\bar{g}_{\mu\nu}$. As we shall see this will not lead to consistent cosmology as we found in the case of cosmological constant. We shall then propose a suitable modification by demanding consistency with cosmological evolution. The action may be written as

$$S = \int d^4x \sqrt{-\bar{g}} \left[\frac{1}{2} \chi^2 \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \chi^4 \phi^2 \right], \quad (46)$$

where, as before, the matter action is same as that in the Einstein's gravity. We have

$$T_\chi = -\chi \langle \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \rangle + 2\chi^3 \langle m^2 \phi^2 \rangle, \quad (47)$$

$$T_{\mu\nu} = \chi^2 \langle \partial_\mu \phi \partial_\nu \phi \rangle. \quad (48)$$

As in the case of radiation we again have,

$$\langle \partial_0 \phi \partial_0 \phi \rangle = \chi^2 \rho, \quad (49)$$

where ρ now stands for the energy density of the non-relativistic matter. Furthermore we have $\langle m^2 \phi^2 \rangle = \rho$. We again solve the equations with $\bar{g}_{\mu\nu}$ equal to the Minkowski metric. The equation of motion for χ , Eq. 17, gives

$$2\xi \frac{d^2 \chi}{d\eta^2} = \kappa \chi^3 \rho. \quad (50)$$

The modified Einstein's equation, Eq. 16, gives,

$$-\frac{2}{\chi} \frac{d^2 \chi}{d\eta^2} + (\xi - 2) \frac{1}{\chi^2} \left(\frac{d\chi}{d\eta} \right)^2 = \frac{\kappa}{2} \chi^2 \rho \quad (51)$$

We eliminate ρ between this equation and Eq. 50 to obtain,

$$(2 + \xi) \chi \frac{d^2 \chi}{d\eta^2} = (\xi - 2) \left(\frac{d\chi}{d\eta} \right)^2. \quad (52)$$

This gives,

$$\frac{d\chi}{d\eta} \propto \chi^\alpha, \quad (53)$$

where $\alpha = (\xi - 2)/(\xi + 2)$. As expected, this gives the standard evolution for non-relativistic matter if we set $\xi = 6$. However the evolution changes for $\xi \neq 6$. This is in contrast to the result we obtained in the case of vacuum dominated or radiation dominated universe. The evolution is well defined as long as $\xi > 2$. The evolution of the energy density in this case is given by,

$$\rho \propto \frac{\chi^{2\alpha}}{\chi^4} \quad (54)$$

For $\xi = 6$, this gives the standard $1/\chi^3$ behavior. However we find a different behavior for $\xi \neq 6$.

Our solution above shows that for $\xi \neq 6$, ρ does not decay as $1/\chi^3$. However we would have expected this behaviour since our matter action is exactly the same as in the case of standard Big Bang model based on covariant gravity. We next explicitly determine the dependence of ρ on χ in the non-relativistic limit directly from our action. As in the case of radiation field we work in the adiabatic limit and assume that χ is a slowly varying function of η . We now rescale our field such that $\bar{\phi} = \chi \phi$. We also define the time varying mass $\bar{m} = \chi m$. In terms of these variables, we may write the action as,

$$S \approx \int d^4x \sqrt{-\bar{g}} \left[\frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} - \frac{1}{2} \bar{m}^2 (\bar{\phi})^2 \right], \quad (55)$$

Here we have ignored derivatives of χ since we assume adiabaticity. The correction terms are proportional to H^2/m^2 , where H is the Hubble constant. Hence, for a wide range of values of m , these can be safely neglected. In terms of $\bar{\phi}$ and \bar{m} the action is same as the standard free

scalar field action in flat space-time. Hence we can directly use known results for this theory. Here we are interested in the extreme non-relativistic limit and hence the time dependence of free field solutions is given by, $\bar{\phi} \sim e^{-i\bar{m}\eta}$. The expectation value of the Hamiltonian density \mathcal{H} in any energy eigenstate is equal to E/V where $E = N\bar{m}$ is the energy corresponding to that state, N is the number of particles and V the spatial volume. The expectation value of \mathcal{H} gets equal contribution from the kinetic energy and potential energy terms. Hence we find,

$$\langle \mathcal{H} \rangle \approx \left\langle \frac{\partial \bar{\phi}}{\partial \eta} \frac{\partial \bar{\phi}}{\partial \eta} \right\rangle \propto \frac{\bar{m}}{V} \propto \chi \quad (56)$$

This implies that

$$\left\langle \frac{\partial \phi}{\partial \eta} \frac{\partial \phi}{\partial \eta} \right\rangle \propto \frac{1}{\chi} \quad (57)$$

Finally using Eq. 49 we find that $\rho \propto 1/\chi^3$. Hence we find that the dependence of ρ on χ , given by Eq. 54, is inconsistent with the expected behaviour if $\xi \neq 6$. This implies that we do not obtain a consistent cosmic evolution for $\xi \neq 6$, if we assume the usual form of non-relativistic matter.

3.5 Generalized Non-relativistic Matter

We next determine whether it is possible to generalize the action such that it leads to consistent cosmology in the non-relativistic limit. We propose a modified free scalar field action of the form

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \chi^2 \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \chi^{2+2\zeta} \phi^2 \right], \quad (58)$$

where ζ is a parameter which we will fix by demanding consistency with cosmic evolution. Following the steps leading to Eq. 57 in the previous section, we find in the present case

$$\left\langle \frac{\partial \phi}{\partial \eta} \frac{\partial \phi}{\partial \eta} \right\rangle \propto \frac{1}{\chi^{2-\zeta}} \quad (59)$$

Hence we find,

$$\rho = \frac{\rho_0}{\chi^{4-\zeta}} \quad (60)$$

where ρ_0 is a constant. We shall fix the value of ζ by demanding consistency with cosmic evolution. We point out that the particle trajectory equation, Eq. 15, will get modified in the present case for massive particles. The third term in Eq. 15 which involves derivatives of χ will change since the mass term in the matter action is different from the standard covariant action. However it will remain unchanged for massless particles. Hence the calculation of the luminosity distance, discussed below, remains the same as in the case of standard covariant theory.

The equation of motion for χ gets modified to

$$2\xi \frac{d^2 \chi}{d\eta^2} = \kappa \zeta \chi^3 \rho. \quad (61)$$

whereas the modified Einstein equation remains the same as Eq. 51. We find a consistent solution to these two equations only if

$$\zeta = \frac{\xi - 4}{2} \quad (62)$$

The resulting solution in terms of cosmic time t ($\chi d\eta = dt$) is found to be,

$$\chi = [1 - \beta C_1(t_0 - t)]^{1/\beta} \quad (63)$$

where $\beta = (12 - \xi)/4$ and

$$C_1^2 = \frac{\kappa \rho_0}{\xi}, \quad (64)$$

Here we have set t_0 as the current time such that $\chi(t_0) = 1$. The luminosity distance in the present model is found to be

$$d_L = \frac{(1+z)}{(\beta-1)C_1} \left[1 - \frac{1}{(1+z)^{(\beta-1)}} \right] \quad (65)$$

where z is the redshift. A fit to the high z supernova data including only such non-relativistic matter leads to $\xi = 9.52$ with $\chi^2 = 549.2$. The Hubble constant in this case is found to be 69.4 Km/(sec Mpc). Hence in this case also we find a good fit to data purely with non-relativistic matter, since χ^2 per degree of freedom less than unity. The energy density ρ in this case falls as $1/\chi^{2.69}$.

4 Schwarzschild Solution

An important check on the parameter ξ is to test whether it leads to a modification of the standard spherically symmetric Schwarzschild solution in vacuum. We address this question in this section. Imposing the unimodular constraint on the metric $\bar{g}_{\mu\nu}$ we can write it as

$$\bar{g}_{\mu\nu} = \text{diag} \left[\frac{1}{A(r)}, -A(r), -r^2, -r^2 \sin^2 \theta \right]. \quad (66)$$

The full metric is given in Eq. 1 with $\chi = \chi(r)$. The determinant \bar{g} is equal to the determinant of the Lorentz metric. The curvature tensor satisfies the following equation in vacuum,

$$-\chi^2 \left[\bar{R}_{\mu\nu} - \frac{1}{4} \bar{g}_{\mu\nu} \bar{R} \right] - \left[(\chi^2)_{;\mu;\nu} - \frac{1}{4} \bar{g}_{\mu\nu} (\chi^2)_{;\lambda}^{\lambda} \right] + \xi \left[\partial_\mu \chi \partial_\nu \chi - \frac{1}{4} \bar{g}_{\mu\nu} \partial^\lambda \chi \partial_\lambda \chi \right] = 0. \quad (67)$$

The equation of motion for χ may be written as

$$\chi \bar{R} = -\xi \bar{g}^{\mu\nu} \chi_{;\mu;\nu}. \quad (68)$$

We have

$$\frac{\bar{R}_{rr}}{A} + A \bar{R}_{tt} = 0. \quad (69)$$

Hence we get

$$\frac{1}{A} (\chi^2)_{;r;r} - \frac{\xi}{A} \partial_r \chi \partial_r \chi + A (\chi^2)_{;t;t} = 0. \quad (70)$$

This leads to the equation

$$\chi \frac{d^2 \chi}{dr^2} - \frac{\xi - 2}{2} \left(\frac{d\chi}{dr} \right)^2 = 0. \quad (71)$$

This implies that

$$\frac{d\chi}{dr} = C \chi^\alpha, \quad (72)$$

where C is a constant and $\alpha = (\xi - 2)/2$. We impose the boundary condition that

$$\frac{d\chi}{dr} \rightarrow 0 \quad (73)$$

as $r \rightarrow \infty$. We also require that $\chi \neq 0$ as $r \rightarrow \infty$. This boundary condition implies that the constant $C = 0$. Hence $d\chi/dr = 0$ and χ is equal to a constant which we may set equal to unity. From Eq. 68 we find that $\bar{R} = 0$. Hence Eq. 67 gives $\bar{R}_{\mu\nu} = 0$. It is clear that this will lead to the standard equation for A and give the standard Schwarzschild solution.

The equation for particle trajectory, Eq. 15, is also the same as in the case of the Einstein's gravity both for massless and massive particles. This is because χ is constant in the present case. Hence the predictions of the standard Schwarzschild solution for spherically symmetric systems remains the same in our theory. In particular we shall obtain the standard Newtonian potential. Furthermore the standard tests of Einstein's theory, such as the perihelion shift of Mercury, would be preserved in this theory. We point out that in the present case the third term on the left hand side of Eq. 15 will have no influence since χ is constant. Hence even if we allow an arbitrary parameter multiplying this term, it will not change the predictions of the standard Schwarzschild solution.

5 Discussion and Conclusions

We have discussed a class of models which respect only unimodular GCI but not the full GCI. In general such models may be constructed by introducing many new parameters. In the present paper we restrict ourselves to a minimal extension by introducing just one parameter, ξ , in the gravitational sector. For $\xi = 6$, the model reduces to Einstein's gravity. We explore a range of values of this parameter such that the model leads to acceptable cosmic evolution. We find that for a wide range of this parameter, $\xi > 2$, the cosmic evolution remains unchanged in the radiation dominated epoch. However we do not find a consistent solution in the case of vacuum dominated Universe for $\xi \neq 6$. Hence if ξ is different from 6, a large cosmological constant does not imply a rapid expansion. In fact the only consistent solution for $\xi \neq 6$ is that the scale factor is zero, if other components, such as matter and radiation, are assumed to be absent. We propose a generalized cosmological constant which does lead to consistent cosmology. In this case we find, remarkably, that a good fit to high redshift supernova data is obtained purely in terms of the generalized cosmological constant without the need for any other components such as non-relativistic dark matter. For the case of non-relativistic matter dominated Universe also we do not obtain a consistent cosmic evolution if $\xi \neq 6$. Here again we propose a generalized mass term in the matter field action so as to obtain a consistent cosmic evolution. In this case also we find that a good fit to the supernova data can be obtained in terms of a single component, namely non-relativistic matter, without requiring cosmological constant. The fit may be further improved by adding an additional component such as a cosmological constant. However since χ^2 per degree of freedom is less than unity purely with non-relativistic matter, there is no motivation for such an additional component. This is particularly interesting since in this model we may consistently set the cosmological constant, including the generalized cosmological constant defined in the present paper, identically equal to zero and yet obtain consistent cosmology. We have also shown that the model admits the standard Schwarzschild solution. Hence it respects the standard tests of the Einstein's gravity on the scale of the solar system.

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